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VI.—*On the Rotation of a Solid Body round a fixed Point; being an Account of the late* Professor MAC CULLAGH'S *Lectures on that Subject. Compiled by the* Rev. SAMUEL HAUGHTON, *Fellow and Tutor of Trinity College, Dublin.*

Read April 23, 1849.

THE following Essay, on the Rotation of a Solid Body round a fixed Point, has been compiled from my notes of Professor MAC CULLAGH'S lectures, delivered in the Hilary Term of the year 1844, in Trinity College. A short account of some of the results contained in it was published by Professor MAC CULLAGH himself, in the Proceedings of the Royal Irish Academy.* As it has appeared to many of Mr. MAC CULLAGH'S friends desirable that a somewhat more detailed account of his researches in this subject should be published, I have, in accordance with this desire, drawn up and presented to the Academy the following account of his lectures on Rotation. I have endeavoured to arrange the subject in a systematic order, and to give the results proved by him during the course of the lectures, carefully excluding all theorems and proofs of theorems, which were not originally given by him, as here stated.

SAMUEL HAUGHTON.

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* Vol. ii. pp. 520, 542.

I.—*Composition of Rotations.*

Let O be the intersection of two axes of rotation, OR, OR'; and let the magnitudes of the rotations be represented by ω, ω'; then the motion impressed upon the body by these two rotations will be the same as the motion produced by a single rotation round an axis, which is represented in magnitude and position by the diagonal of the parallelogram formed by ω, ω'. For, draw through any point I of the body a plane perpendicular to the line OI, and project upon this plane the parallelogram formed by ω, ω';—the sides of this parallelogram will be ω sin ROI and ω' sin R'OI. Now the velocities impressed upon the point I by the rotations ω and ω', are OI.ω sin ROI, and OI.ω' sin R'OI; and the directions of these velocities are perpendicular to the sides of the projected parallelogram. Hence, if this parallelogram be turned in its plane through 90°, its sides will represent in magnitude and direction the actual velocities; the resultant of these velocities is perpendicular to the projection of the diagonal of the parallelogram (ω, ω'); this projection, turned round through 90°, will represent the actual velocity, which is therefore the same in magnitude and direction as would be produced by a single rotation represented by the diagonal of (ω, ω'). Hence rotations may be resolved along three rectangular axes by the same laws as couples, and they must be counted positive when the motion produced is from z to x, x to y, y to z, and *vice versâ.*

II.—*Linear Velocities produced by a given Rotation.*

Let the origin of co-ordinates be assumed on the axis of rotation, and let the magnitude of the rotation and of its components be represented by (ω, p, q, r): the velocity of any point (x, y, z) is in a direction perpendicular to the plane containing the axis of rotation and the point (x, y, z); and its magnitude is represented by the area of the triangle whose angles are situated at the origin, the point (x, y, z), and the point (p, q, r). Hence, the components of the linear velocity are represented by the projections of this triangle on the co-ordinate planes. These projections are

$$\begin{aligned} u &= qz - ry; \\ v &= rx - pz; \\ w &= py - qx. \end{aligned} \tag{1}$$

III.—*To represent geometrically the Moments of Inertia of a Body with respect to Axes drawn through a fixed Point.*

The moment of inertia of a body with respect to any axis (α, β, γ) is

$$M = A' \cos^2 \alpha + B' \cos^2 \beta + C' \cos^2 \gamma - 2L' \cos \beta \cos \gamma - 2M' \cos \alpha \cos \gamma - 2N' \cos \alpha \cos \beta;$$

where

$$\begin{aligned} A' &= \int (y^2 + z^2)\, dm, & L' &= \int yzdm; \\ B' &= \int (x^2 + z^2)\, dm, & M' &= \int xzdm; \\ C' &= \int (x^2 + y^2)\, dm, & N' &= \int xydm. \end{aligned}$$

Assume $M = \frac{\mu}{r^2}$; μ being the mass of the body, and r a distance measured on the line (α, β, γ) and construct the ellipsoid whose equation is

$$A'x^2 + B'y^2 + C'z^2 - 2L'yz - 2M'xz - 2N'xy = \mu; \tag{2}$$

then it is evident that the moments of inertia of the body with respect to axes passing through the fixed point are represented by the squares of the reciprocals of the radii vectores of this ellipsoid. Assume $A = \mu a^2$, $B = \mu b^2$, $C = \mu c^2$, and let the axes of co-ordinates be the axes of the ellipsoid; its equation will thus become

$$a^2x^2 + b^2y^2 + c^2z^2 = 1; \tag{3}$$

and the equation of the reciprocal ellipsoid will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{4}$$

This latter ellipsoid may be called the ellipsoid of gyration, as the perpendiculars on its tangent planes represent the radii of gyration; this is evident from the consideration, that these perpendiculars are reciprocal to the radii vectores of the ellipsoid (3). In fact, the moment of inertia with respect to any axis will be represented by the formula

$$M = (a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma)\,\mu = \mu P'^2 = \frac{\mu}{R'^2}; \qquad (5)$$

(R, P) denoting the radius vector and perpendicular on tangent plane of the ellipsoid (4); and $(R'P')$, the corresponding lines in the ellipsoid (3).

IV.—*To find the Magnitude, Position, and Direction of the Statical Couple produced by the Centrifugal Forces.*

If from any point (x, y, z) of the body, a perpendicular be let fall on the axis of rotation (α, β, γ), the centrifugal force will be represented by the product of the square of the angular velocity and this perpendicular; the corresponding elementary statical couple will be found by multiplying the centrifugal force by the distance from the foot of the perpendicular to the origin, which is represented by the quantity $(x \cos \alpha + y \cos \beta + z \cos \gamma)$. The components of the elementary couple will be proportional to the projections of the triangle formed by the lines before mentioned. The components of the elementary couples must be integrated for the entire extent of the body, and the integrals thus found will be the components of the couple produced by centrifugal force: the expressions are as follows:

$$\omega^2 (x \cos \alpha + y \cos \beta + z \cos \gamma)(z \cos \beta - y \cos \gamma)\, dm;$$
$$\omega^2 (x \cos \alpha + y \cos \beta + z \cos \gamma)(x \cos \gamma - z \cos \alpha)\, dm;$$
$$\omega^2 (x \cos \alpha + y \cos \beta + z \cos \gamma)(y \cos \alpha - x \cos \beta)\, dm.$$

If the axes of co-ordinates be principal axes, these expressions, when integrated, will become

$$\begin{aligned} \omega^2 \cos \beta \cos \gamma\, (B - C) &= qr\, (B - C); \\ \omega^2 \cos \alpha \cos \gamma\, (C - A) &= pr\, (C - A); \qquad (6) \\ \omega^2 \cos \alpha \cos \beta\, (A - B) &= pq\, (A - B): \end{aligned}$$

p, q, r being the components of the angular velocity ω. The position of the resultant couple may be expressed by means of the ellipsoid (4). If a tangent plane be drawn to this ellipsoid at the point (x, y, z), and perpendicular to the line (α, β, γ), it may be easily shown that the projections of the triangle formed by the radius vector and perpendicular are represented by the quantities

$$\cos\beta\cos\gamma\,(b^2-c^2),\quad \cos\alpha\cos\gamma\,(c^2-a^2),\quad \cos\alpha\cos\beta\,(a^2-b^2)\,;$$

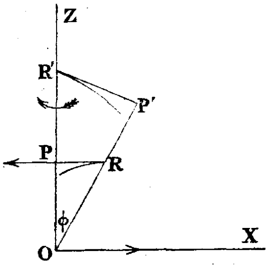
these three expressions multiplied by $\mu\omega^2$ will produce the quantities used in (6). Hence it appears, that the couple produced by the centrifugal forces lies in the plane of the radius vector and perpendicular to a tangent plane of the ellipsoid (4); the tangent plane being perpendicular to the axis of rotation. Also, the magnitude of the resultant couple is proportional to the triangle formed by the radius vector and perpendicular.

The differential equations of motion commonly used in the solution of this problem may be deduced immediately from equations (6). In fact, as the axes of co-ordinates are axes of permanent rotation, the increment of angular velocity round each axis will be equal to the statical couple of the applied forces (including centrifugal forces), divided by the moment of inertia round that axis; the statement of this fact, in analytical language, will give the equations of motion:

$$\begin{aligned} A\frac{dp}{dt} &= (B-C)\,qr+L\,; \\ B\frac{dq}{dt} &= (C-A)\,pr+M\,; \qquad (7) \\ C\frac{dr}{dt} &= (A-B)\,pq+N: \end{aligned}$$

(L, M, N) being the components of the applied statical couple.

The *position* and *magnitude* of the couple produced by the centrifugal forces are easily found by the method which has been just given; but the *direction* will be found more readily by taking more particular axes of co-ordinates. Let the axis of rotation be the axis OZ, and the plane of radius vector and perpendicular be the co-ordinate plane XOZ. In the accompanying figure OR′ and OP′ are the radius vector and perpendicular of the ellipsoid (2), and OR, OP the radius vector and perpendicular of the ellipsoid (4), which is reciprocal to the former; the rotation is positive, in the direction indicated by the arrow. As the rotation is round the axis of z, it is easy to see that the statical couple produced by centrifugal force



will have for components, round the axes of x and y respectively, the quantities $\omega^2\int yzdm$, $\omega^2\int xzdm$ taken with their proper sign; i. e. the components are $\pm\,\omega^2 L'$, $\pm\,\omega^2 M'$; $L'M'$, being coefficients in the equation of the ellipsoid

$$A'x^2 + B'y^2 + C'z^2 - 2L'yz - 2M'xz - 2N'xy = \mu.$$

The tangent plane to this ellipsoid, applied at the point (x, y, z) will be

$$(A'x - M'z - N'y)\,x' + (B'y - N'x - L'z)\,y' + (C'z - L'y - M'x)\,z' = \mu.$$

At the point R′ the tangent plane will be perpendicular to the plane XOZ, and will be found by making $x = 0$, $y = 0$, and destroying the coefficient of y' in the preceding equation. These conditions give us $L' = 0$, which proves that the statical couple produced by centrifugal force lies altogether in the plane XOZ. The equation of the tangent plane is the same as the equation of the line R′P′, and is

$$C'z' - M'x' = \frac{\mu}{z}\,.$$

Hence we obtain

$$\tan\phi = -\frac{M'}{C'}\,.$$

The value of the centrifugal couple is $\omega^2 M'$, which is found from the preceding equation by replacing C' and $\tan\phi$ by their values μP^2, and $\frac{Q}{P}$; Q being the line RP.

We thus obtain finally the centrifugal couple lying in the plane XOZ, and expressed by the equation

$$\omega^2\int xzdm = -\,\mu\omega^2 PQ. \qquad (8)$$

It thus appears that the centrifugal couple lies in the plane of radius vector and perpendicular, is proportional to the area of the triangle ROP, and has a direction opposite to the direction of rotation.

V.—*To find the Relation between the Plane of principal Moments and the Axis of Rotation at any Instant.*

The motion of the body at any instant consists of a rotation of a certain magnitude round a certain axis; this rotation might be produced by an impulsive couple of a determinate magnitude and direction. The statical impulsive couple thus conceived is the couple of principal moments. Let this couple be represented by G, and act round the axis OR (fig. 1, p. 148); then the corresponding axis of rotation will be the perpendicular OP, and the relation between G and ω may be thus found. Let the axes of co-ordinates be the axes of the ellipsoid (4), the radius vector being determined by the angles (λ, μ, ν), and the axis of rotation by the angles (α, β, γ). From mechanical considerations we obtain the equations

$$G \cos \lambda = Ap = \mu\omega\, a^2 \cos \alpha\,;$$
$$G \cos \mu = Bq = \mu\omega\, b^2 \cos \beta\,;$$
$$G \cos \nu = Cr = \mu\omega\, c^2 \cos \gamma.$$

Hence we obtain

$$\frac{\cos \lambda}{\cos \nu} = \frac{a^2 \cos \alpha}{c^2 \cos \gamma}, \quad \frac{\cos \mu}{\cos \nu} = \frac{b^2 \cos \beta}{c^2 \cos \gamma}, \qquad (9)$$
$$\omega = \frac{G}{\mu PR} = \frac{G \cos \phi}{\mu P^2}.$$

The first two of these equations prove that the axis of rotation is the perpendicular on tangent plane of the ellipsoid, and the last equation gives the magnitude of the rotation in terms of the impressed couple and quantities determined by the nature of the body itself. Equations (9) are true, whatever be the forces acting on the body; if no forces act, G will be fixed in magnitude and position in space, by the principle of conservation of areas, but will change its position in the body, the axis of rotation accompanying it, and changing its position both in the body and in space.

VI.—*Rotation produced by Centrifugal Force; particular Properties of the Motion when no Forces act.*

The axis of rotation produced by the centrifugal couple always lies in the plane of principal moments. This theorem may be thus proved: Let the radius vector and perpendicular be drawn, which coincide with the axis of principal moment and axis of rotation at any instant; a line perpendicular to the plane of radius vector and perpendicular is the axis of centrifugal couple;

this line and the original radius vector are axes of the section of the ellipsoid made by their plane; at the point where the axis of the centrifugal couple pierces the ellipsoid let a tangent plane be applied; the perpendicular let fall on this tangent plane is the axis of rotation produced by centrifugal forces. From the construction it is evident that the plane of the second radius vector and perpendicular is perpendicular to the axis of G; hence the axis of the centrifugal couple and the axis of rotation produced by it, always lie in the plane of principal moment. Two important corollaries follow from the theorem just demonstrated, in the case where no forces act:—First, the component of angular velocity round the axis of primitive impulse is constant during the motion. Secondly, the radius vector which coincides with the axis of G is of constant length during the motion. The first theorem is obvious; for as the axis of rotation produced by centrifugal force is always perpendicular to the axis of G, it cannot alter the rotation round that axis. The second theorem follows from equation (9), from which we deduce

$$\omega \cos \phi = \frac{G}{\mu R^2}. \tag{10}$$

The left hand member of this equation is constant by the preceding theorem; and G is constant, since there is no external force; therefore R is constant.

As the axis of G is fixed in space, and the line R is constant, it is evident that the axis of G will describe in the body the cone of the second degree, determined by the intersection of the ellipsoid (4) with the sphere whose radius is R. The equation of this cone is

$$\frac{R^2 - a^2}{a^2} x^2 + \frac{R^2 - b^2}{b^2} y^2 + \frac{R^2 - c^2}{c^2} z^2 = 0. \tag{11}$$

As the axis of principal moments describes this cone in the body, it is accompanied by the axis of rotation, which is always the corresponding perpendicular on tangent plane of the ellipsoid. The cone described by the axis of rotation might be found thus. Let tangent planes be applied to the ellipsoid along the spherical conic in which the cone (11) cuts the ellipsoid. From the centre let fall perpendiculars on these tangent planes; the locus of these perpendiculars is the required cone.

VII.—*The Axis of principal Moments is fixed in Space.*

This is evident from D'ALEMBERT'S principle, but may be shown by geometrical considerations in the particular case under consideration. The axis varies in position in the body, in consequence of the centrifugal couple, which must be compounded with the impressed couple at each instant. Referring to equation (8), the value of the centrifugal couple is $-\mu\omega^2 PQdt$, the principal moment being $G = \mu\omega PR$ (vid. 9). Hence the angle through which the axis of principal moment shifts in an element of time is $-\dfrac{\omega Qdt}{R}$; this angle, multiplied by the constant radius vector, will give the elementary motion on the spherical conic traced by the axis of principal moment on the surface of the ellipsoid; this motion is therefore $-\omega Qdt$; but in the same time the point of the body which coincides with the point where the axis of moments pierces the spherical conic will describe the angle $+\omega Qdt$ in consequence of the angular rotation. Hence the axis of moments will remain fixed in space, and will move in the body with a velocity proportional to the tangent of the angle between the radius vector and perpendicular, the motion being in a direction opposite to the direction of the rotation; this is evident from the consideration that $Q\omega = P\omega \tan\phi$, $P\omega$ being constant and equal to $\dfrac{G}{\mu R}$ (vid. 9).

VIII.—*To find the Motion of the principal Axis in the Body.*

First Method.

The point of the principal axis of moments, which is situated at the distance R from the centre, moves on the spherical conic which has been determined. Let this point be projected on the three co-ordinate planes; then, since the spherical conic is projected into a conic section, the movement of the axis of moments is reduced to the movement of a point on a conic section, according to a law which must be determined. The radius vector describes an elementary triangle in the surface of the cone (11); let the projections of this triangle on the co-ordinate planes be (dA_1, dA_2, dA_3); we obtain easily

$$\frac{dA_1}{dt} = y\frac{dz}{dt} - z\frac{dy}{dt}, \quad \frac{dA_2}{dt} = z\frac{dx}{dt} - x\frac{dz}{dt}, \quad \frac{dA_3}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}.$$

Substituting in these equations the values of the velocities given by (1), we obtain

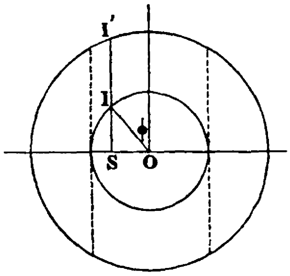
$$\begin{aligned} \frac{dA_1}{dt} &= P\omega\,\frac{R^2 - a^2}{a^2}\,x = (R^2 - a^2)\,p\,; \\ \frac{dA_2}{dt} &= P\omega\,\frac{R^2 - b^2}{b^2}\,y = (R^2 - b^2)\,q\,; \\ \frac{dA_3}{dt} &= P\omega\,\frac{R^2 - c^2}{c^2}\,z = (R^2 - c^2)\,r. \end{aligned} \qquad (12)$$

These equations prove, that the *areolar velocity* of the projection on a co-ordinate plane varies at the ordinate to that plane. By means of the method of quadratures, we may determine from equations (12) the position of the projections of the principal axis at any instant, and hence deduce the position of the axis itself.

Second Method.

If the spherical conic be projected on a cyclic plane of the ellipsoid of gyration, by lines parallel to x and z, the projections will be two concentric circles, and the corresponding projections will lie on the same ordinate SII′ (fig. 2). The inner circle will belong to the projection parallel to x, if R be greater than b, and will belong to the projection parallel to z if R be less than b; and if R be equal to b, the two circles will coincide with each other and with the spherical conic, which in this case becomes the circular section of the ellipsoid. The projected point will revolve round the circumference of the inner circle, and will vibrate on the circumference of the outer circle, between the dotted lines. It is evident that the mean axis of the ellipsoid OY lies in the plane of the figure. Let SI and SI′ be equal to ρ, ρ', and let C, C' denote the radii of the two circles: the velocities of the projections in the circles will evidently be

FIG. 2.



$$V = \frac{C}{\rho}\frac{dy}{dt}, \quad V' = \frac{C'}{\rho'}\frac{dy}{dt};$$

C and C' having the values

$$C = b\sqrt{\left(\frac{a^2 - R^2}{a^2 - b^2}\right)}, \quad C' = b\sqrt{\left(\frac{R^2 - c^2}{b^2 - c^2}\right)}.$$

The value of $\frac{dy}{dt}$ deduced from (1) is,

$$\frac{dy}{dt} = P\omega\left(\frac{1}{c^2} - \frac{1}{a^2}\right) xz = P\omega\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\rho\rho' \sin\theta\cos\theta;$$

θ being the angle made by the plane of the circular section with the plane (x, y).

$$\sin\theta = \frac{c}{b}\sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad \cos\theta = \frac{a}{b}\sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

Introducing these values of $\frac{dy}{dt}$, $\sin\theta$ and $\cos\theta$, and for $P\omega$ its value $\frac{G}{\mu R}$, we obtain finally for the velocities

$$\begin{aligned} V &= \frac{G}{\mu}\sqrt{\left\{\left(\frac{1}{R^2} - \frac{1}{a^2}\right)\left(\frac{1}{c^2} - \frac{1}{b^2}\right)\right\}}\,\rho' = K\rho'; \\ V' &= \frac{G}{\mu}\sqrt{\left\{\left(\frac{1}{R^2} - \frac{1}{c^2}\right)\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right\}}\,\rho = K'\rho. \end{aligned} \tag{13}$$

The velocity of each projection, therefore, varies as the ordinate of the other. This theorem enables us to find a simple expression for the time. Using the angle (ϕ) marked in fig. 2, we obtain

$$\frac{Cd\phi}{dt} = K\sqrt{(C'^2 - C^2\sin^2\phi)};$$

(ϕ, C, K) belonging to the projection parallel to axis of x. If (ψ, C', K') be the corresponding quantities for the other projection, we obtain also

$$\frac{C'd\psi}{dt} = K'\sqrt{(C^2 - C'^2\sin^2\psi)};$$

or, since it is easily seen that $\frac{K}{K'} = \frac{C}{C'}$, we obtain finally

$$K'dt = \frac{d\phi}{\sqrt{\left(1 - \frac{C^2}{C'^2}\sin^2\phi\right)}};$$
$$Kdt = \frac{d\psi}{\sqrt{\left(1 - \frac{C'^2}{C^2}\sin^2\psi\right)}}. \qquad (14)$$

The motion of the principal axis of moments is, therefore, expressed by an elliptic function of the first kind.

The motion of the axis of moments is determined by the magnitude of the radius vector of the ellipsoid, which is the axis of the original couple impressed upon the body; if this radius vector be greater than the mean axis of the ellipsoid, the corresponding spherical conic will have the axis of x for its internal axis; and if the radius be less than the mean axis, the axis of z will be the internal axis of the conic; in no case will the mean axis be the internal axis of the spherical conic. If the radius R be nearly equal to either the greatest or least semi-axis, the expression (14) for the time may be integrated. Let R be nearly equal to the greatest semi-axis. The first of the equations (14) belongs to the interior circle, which is of small dimensions in the case supposed; the second equation expresses the vibratory motion of the projection, through a small arc of the outer circle, which will have a radius much greater than the inner circle; we may, therefore, suppose the angle ψ to be equal to its sine. Multiplying both sides of the equation by $\frac{C'}{C}$ we obtain

$$\frac{C'K}{C}dt = K'dt = \frac{\frac{C'}{C}d\psi}{\sqrt{\left(1 - \frac{C'^2}{C^2}\psi^2\right)}}.$$

Hence

$$\frac{C'\psi}{C} = \sin(K't + A). \qquad (15)$$

If T_0 denote the time of a complete oscillation or revolution of axis of moments about the axis of x, and T_r the time of a revolution of the body round the axis of x, the following relation between these two periods may be readily deduced from (15):

$$T_0 = T_r \frac{bc}{\sqrt{\{(a^2 - b^2)(a^2 - c^2)\}}}. \qquad (16)$$

If the axis of moments, and consequently the axis of revolution, be situated near the axis of greatest or least inertia, it will always continue near this axis ; if, however, it be situated near the mean axis, the movement of the body will be determined by the following construction. Let the two cyclic planes of the ellipsoid be drawn through the mean axis; they will divide the ellipsoid into two regions, in one of which is situated the axis of maximum inertia, and in the other the axis of minimum inertia. The spherical conic described by the axis of principal moments will have the first or second of these axes for its internal axis, according as R is greater or less than the mean axis. If the axis of principal moments lie in one of the cyclic planes, the spherical conic becomes a circle, and its two projections become identical with itself (fig. 2, p. 148); the expressions (14) are reduced to the form

$$Kdt = \frac{d\phi}{\cos\phi};$$

which when integrated gives

$$Kt + A = \log \cot\left(\frac{\pi}{4} - \frac{\phi}{2}\right);$$

or,

$$\cot\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \cot\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\epsilon^{Kt}; \qquad (17)$$

ϕ_0 being the value of ϕ corresponding to $t = 0$, and K being expressed by the following quantity:

$$K = \frac{G}{\mu}\frac{\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}}{b^2ac}.$$

It is evident from the equation (17), that the axis of moments will coincide with the mean axis of inertia at the end of an infinite time.

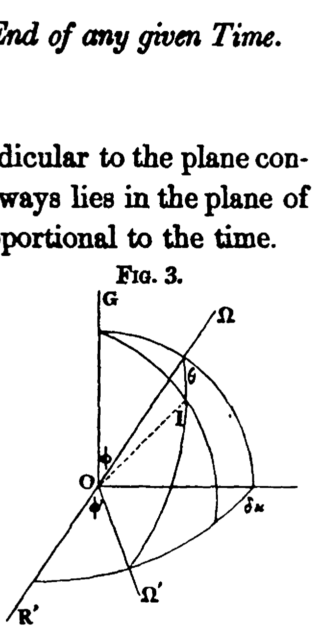
IX.—*To find the Position of the Body in Space at the End of any given Time.*

First Method.

The radius vector of the ellipsoid, which is perpendicular to the plane containing the axes of principal moment and of rotation, always lies in the plane of principal moment, and describes in that plane areas proportional to the time.

Let OG, OΩ be the axes of principal moment and of rotation; OR′, OΩ', the axes of centrifugal couple and of corresponding rotation; the plane ΩOΩ' will contain the two successive positions of the axis of rotation. Let OI be the position of the axis of rotation at the end of the time δt; then δu will be equal to the angle described in the fixed plane by the line OR′. Let R' and P' be the radius vector and perpendicular corresponding to the centrifugal couple and its axis of rotation. The following relations are evident from the figure

FIG. 3.



$$\frac{\omega}{\omega'} = \frac{\sin \Omega'\mathrm{OI}}{\sin \Omega\mathrm{OI}} = \frac{\cos \phi'}{\sin \phi\delta u}; \text{ because } \sin \Omega'\mathrm{OI} = \frac{\cos \phi'}{\sin \theta}, \sin \Omega\mathrm{OI} = \frac{\sin \phi\delta u}{\sin \theta};$$

but from mechanical considerations,

$$\frac{\omega}{\omega'} = -\frac{P'R'}{PR\omega \sin \phi\delta t}; \text{ because, } \omega = \frac{G}{\mu PR}, \quad \omega' = -\frac{G\omega \sin \phi\delta t}{\mu P'R'}.$$

Hence, by equating the geometrical and mechanical expressions, we obtain

$$-R'^2\delta u = \omega PR\delta t = \frac{G}{\mu}\delta t. \tag{18}$$

The position of the body in space is thus reduced to quadratures; but the problem may be solved more readily in the following manner.

Second Method.

The axis of principal moments, appearing to move in a direction opposite to the rotation, describes in the body the cone whose equation has been given (11).

If the cone reciprocal to this cone be described, one of its sides will lie in the fixed plane, and the whole motion of the body in space will be the same as the motion of this cone, which partly slides and partly rolls on the fixed plane, the sliding motion being uniform. This theorem is evident by resolving the angular velocity ω into two components, one round the axis of principal moments, and the other in a direction perpendicular to this, round the side of the reciprocal cone, which is in contact with the fixed plane. These components are $\omega \cos \phi$ and $\omega \sin \phi$; $\omega \cos \phi$ being constant and producing the sliding motion, while $\omega \sin \phi$ represents the angular velocity round the side of the cone in contact with the fixed plane. The angle described by the side of the reciprocal cone in the fixed plane at the end of a given time, is, therefore, the algebraic sum of two angles, one of which is proportional to the time, and the other is the angle described in the cone in consequence of the rotation $\omega \sin \phi$, and is, therefore, measured by the arc of a spherical conic. The position of the body at the end of the time t is thus found:—determine by equation (14) the position of the axis of principal moments in the cone (11); the corresponding position of the component axis of rotation in the reciprocal cone is therefore known. Hence the angle described in the time t in the fixed plane is

$$\Theta = \int \omega \cos \phi dt \pm \int \frac{ds}{R} = \omega \cos \phi . t \pm \frac{s}{R}. \tag{19}$$

The equation of the reciprocal cone is

$$\frac{a^2x^2}{R^2 - a^2} + \frac{b^2y^2}{R^2 - b^2} + \frac{c^2z^2}{R^2 - c^2} = 0. \tag{20}$$

In (19) the positive or negative sign must be used according as R is less or greater than the mean axis of the ellipsoid; this is evident from the composition of rotations, and from the consideration that in the former case the axis of rotation falls inside the cone (11), while in the latter case it falls outside.

X.—*To find a Point in a given Axis of Rotation, which being fixed, the Axis will be permanent.*

Let R′R″ (fig. 4) be the given axis, round which the body revolves with a rotation expressed by ω; describe the ellipsoid of gyration round the centre of gravity O, and draw OP′ parallel to R′R″. The centrifugal force $\omega^2 rdm$ at any point (x, y, z), may be resolved into two components, $\omega^2 \rho dm$ and $\omega^2 . R'P' . dm$; r and ρ denoting the distances of the point from the axes R′R″ and OP′ respectively; the effect of the rotation round R′R″ is therefore the same as an equal rotation round OP′, together with a number of parallel constant forces applied to each point of the body. The rotation round OP′ produces a centrifugal couple represented by $-\ \mu\omega^2 . OP . PR$ (vid. 8); or, determining the point R′ by thecondition $OP . PR = OP' . P'R'$, the centrifugal couple is $-\ \mu\omega^2 . OP' . P'R'$. The resultant of the parallel forces is a force applied at the centre of gravity, acting in the direction parallel to R′P′, and equal to $\mu\omega^2 . R'P'$. Comparing this with the centrifugal couple, it is evident that the forces at O destroy each other, and, therefore, the total result of the rotation round R′R″ is to produce a force acting at the point R′, which has been just determined. If this point be fixed, the axis R′R″ will be a permanent axis of rotation. The condition by which the point R′ is found is, that the triangle OR′P′ is equal to the triangle ORP; hence, if an ellipsoid confocal to the ellipsoid of gyration be described through the point R′, it will be perpendicular to the line R′R″. The general construction for permanent axes is, therefore, the following. Let the ellipsoid of gyration be described, and confocal ellipsoids; any line which pierces one of these ellipsoids at right angles is a permanent axis of rotation for the point of intersection.

FIG. 4.